Exercices

(1.) $H_3 = \langle x_1, x_2, x_3 \mid x_1^{-1}x_2x_1 = x_2^2,\ x_2^{-1}x_3x_2 = x_3^2,\ x_3^{-1}x_1x_3 = x_1^2 \rangle$ is trivial.

Proof $\quad x_2x_1x_2^{-1} = x_1x_2$; conjugate with x_3 $\Rightarrow$ use $x_3^{-1}x_2x_3 = x_2x_3^{-1}$. $\Rightarrow$

$$\underbrace{x_3^{-1}x_2x_3}_{=\,x_2x_3^{-1}}\ \underbrace{x_3^{-1}x_1x_3}_{=\,x_1^2}\ \underbrace{x_3^{-1}x_2^{-1}x_3}_{=\,x_3x_2^{-1}} = \underbrace{x_3^{-1}x_1x_3}_{=\,x_1^2}\ \underbrace{x_3^{-1}x_2x_3}_{=\,x_2x_3^{-1}}$$

i.e. $$x_2\underbrace{x_3^{-1}x_1^2x_3}_{=\,x_1^4}x_2^{-1} = x_1^2x_2x_3^{-1} \iff x_2x_1^4x_2^{-1} = x_1^2x_2x_3^{-1} \Rightarrow$$

$$x_3 = x_2^{-1}x_1^{-4}x_2x_1^2x_2$$

$\Rightarrow$ the sbgp $H = \langle x_1, x_2\rangle \subset H_3$ coincide with H_3.

But H is a quotient of $G = BS(1,2) = \langle x_1, x_2 \mid x_1^{-1}x_2x_1 = x_2^2\rangle$.

$G \hookrightarrow GL(2,\mathbb{Q})$, $G \simeq \left\langle \begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}\right\rangle$. This is triangular group. Verify that

$$\left.\begin{aligned}\left[\begin{pmatrix}*&*\\0&*\end{pmatrix}, \begin{pmatrix}*&*\\0&*\end{pmatrix}\right] &= \begin{pmatrix}1&*\\0&1\end{pmatrix}\\ \left[\begin{pmatrix}1&*\\0&1\end{pmatrix}, \begin{pmatrix}1&*\\0&1\end{pmatrix}\right] &= 0\end{aligned}\right\} \Rightarrow \quad G'' = [G', G'] = 1. \quad \Rightarrow \quad H'' = 1 \text{ since } G \twoheadrightarrow H.$$

On the other hand $H_3' = H_3$ because abelianization of H_3 is trivial. Hence

$H_3'' = H_3' = H_3 \Rightarrow H_3 = 1$. □

(2) A HNN extension $G*_\theta$, $\theta: H \to K$ is a subgroup of an amalgamated product $G*\mathbb{Z}$ and $G*\mathbb{Z}$.

Pf. $E = G*_\theta = \langle G, t \mid tht^{-1} = \theta(h),\ h \in H\rangle$.

$U = G * \langle u\rangle$, $V = G*\langle v\rangle$

The subgroup C of U generated by G and conjugate uHu^{-1} is $C \cong G * uHu^{-1} \simeq G*H$

The subgroup D of U generated by G and conjugate $v^{-1}Hv$ is $D \simeq G * v^{-1}Kv \simeq G*K$.

$\exists$ an isomorphism $\varphi: C \to D$, $\varphi|_G = id$, $\varphi|_{uHu^{-1}} = v^{-1}\varphi|_H v \in v^{-1}Kv$.

Set then $H = G*\langle u\rangle \underset{C=D}{*} G*\langle v\rangle$.

We have $G \hookrightarrow H$. Set then $\tilde{t} = vu$; then $\Rightarrow$ $\tilde{t}h\tilde{t}^{-1} = \varphi(h)$. Set

$\tilde{E} = \langle G, \tilde{t}\rangle \subset H$. We claim $\tilde{E} \simeq E$. Set $\alpha: E \to \tilde{E}$, $\alpha|_G = id$, $\alpha(t) = \tilde{t}$;

Let $\gamma: H \to E$, $\gamma|_G = id$, $\gamma(u) = 1$ (on U) ; $\gamma|_C = \gamma|_D \Rightarrow \gamma: H \to E$ well def.
$\gamma|_G = id$, $\gamma(v) = t$ (on V)

Set $\beta = \gamma|_{\tilde{E}} \Rightarrow \beta: \tilde{E} \to E$ is the inverse of α.

③ A, B f. presented and $H \subset A$, $H \subset B$. Then $A *_H B$ f. presented iff H is f. generated.

Pf. Let H be infinitely generated. If $A = \langle X \mid R \rangle$, $B = \langle Y \mid S \rangle$ write generators of H as

$H = \langle h_1(x) \dots h_n(x) \dots \rangle = \langle k_1(y), \dots, k_n(y) \dots \rangle$, $h_i(\underline{x}) = k_i(\underline{y})$.

$A *_H B = \langle X, Y \mid R, S, h_i(x) k_i(y)^{-1}, i \in \mathbb{Z}_+ \rangle$.

$X \cup Y$ finite $\Rightarrow$ Neumann thm if $A *_H B$ is f. presented then we can extract f. many relations $\Rightarrow$

$A *_H B = \langle X, Y \mid R, S, h_i(x) k_i(y)^{-1}, i \leq n \} =: G$

The subgroup $\widetilde{H} = \langle h_1(x), \dots, h_n(x) \rangle \subset A$, ~~$\langle k_1(y), \dots, k_n(y) \rangle \subset B$~~ IS NOT all of H

$\Rightarrow \exists$ an element $h(x) \in H - \widetilde{H}$; Write $h(x) = k(y)$ in $A *_H B$, as an element of B. Then also

$k(y) \in H - \widetilde{K}$ where $\widetilde{K} = \langle k_1(x), \dots, k_1(y) \rangle \subset B$.

Moreover G is an amalgamated product $G = A *_{\widetilde{H} = \widetilde{K}} B$. Then the word $h(x) k(y)^{-1}$ is strictly alternating in $G \Rightarrow$ it is not trivial in G. However in $A *_H B$, $h(x) k(y)^{-1} = 1 \Rightarrow$ contradiction.

Neumann thm: Let G be a finitely presented group. Then any presentation of G with f. many generators can be made finite by throwing away all but finitely many relations.

Lemma. $G = \langle x_1, \dots x_m \mid r_1, \dots, r_e \rangle$ and $G = \langle x_1, \dots x_m \mid s_1, \dots \rangle \Rightarrow \exists n$ s.t. $G = \langle x_1, \dots x_m \mid s_1, \dots, s_n \rangle$.

Pf. Normal subgroup of $\mathbb{F}(x_1, \dots x_m)$ generated by r_j's = the one generated by the s_i's. $\Rightarrow$ we can write $r_j = W_j(s_1, \dots)$. But $\underline{W_j}$ are products of conjugates of s_i and $s_i^{-1} \Rightarrow$ only finitely many occurrences $\Rightarrow \langle r_j \rangle = \langle s_i \rangle_{i \leq n}$ □

Tietze moves

(T1) $\langle X \mid R \rangle \longrightarrow \langle X, Y \mid R \cup \{ y\, w_y^{-1}(x), y \in Y \} \rangle$

(T2) $\langle X \mid R \rangle \longrightarrow \langle X, Y \mid R \cup S$, S consequences of $R \rangle$

T1, T2 and their inverses are called Tietze moves.

Tietze Thm: 1) $\langle X \mid R \rangle = \langle Y \mid S \rangle \Rightarrow$ we can pass from first to second by Tietze moves.

2) Finite presentations $\Rightarrow$ only finite Tietze moves (Y, S finite) at each step and finitely many steps.

Pf. i) Group G is quotient $\mathbb{F}(X) \xrightarrow{\varphi} \langle X \mid R \rangle$, $\mathbb{F}(Y) \xrightarrow{\psi} \langle Y \mid S \rangle$. Thus $\psi(y) = w_y(\varphi(x))$ $\forall y \in Y$.

Use T1 $\Rightarrow$ $G = \langle X \cup Y \mid R \cup \{ y w_y(x)^{-1}, y \in Y \} \rangle$.

ii) Each $s \in S$ is a relator $\Rightarrow$ use T2

$G = \langle X \cup Y \mid R \cup \{ y w_y^{-1}(x) \}_{y \in Y} \cup S \rangle$

iii) Moreover $\varphi(x) = v_x(\psi(y))$ $\forall x \in X$ $\Rightarrow$.

$$G = \langle X \cup Y \mid R \cup S \cup \{ y w_y^{-1}(x), y \in Y \} \cup \{ x v_x(y)^{-1}, x \in X \} \rangle$$

(iv) Write $R = \{ r(x), r \in R \} \Rightarrow$ rewrite

$$G = \langle X \cup Y \mid \{ r(x), r \in R \} \cup S \cup \{ y w_y^{-1}(x), y \in Y \} \cup \{ x v_x(y)^{-1}, x \in X \} \cup$$
$$\{ r(v_x(y)), r \in R, y \in Y \} \cup \{ y w_y^{-1}(v_x(y)), y \in Y \}$$

The inverse of T2 $\Rightarrow$

$$G = \langle X \cup Y \mid S \cup \{ r(v_x(y)), r \in R \} \cup \{ y w_y^{-1}(v_x(y)), y \in Y \}$$

v) Use inverse T1 $\Rightarrow$ throw away X

$$G = \langle Y \mid S \cup \{ r(v_x(y)), r \in R \} \cup \{ y w_y^{-1}(v_x(y)), y \in Y \}$$

Use inverse T2 $\Rightarrow$ $G = \langle Y \mid S \rangle$ $\square$

Pf. Neumann thm. Let $G = \langle x_1, \dots, x_k \mid r_1 \dots r_e \rangle$, $G = \langle y_1, \dots y_m ; s_1, s_2, \dots \rangle$.

Add first y_i's to generators and then remove x_j's:

$$G = \langle x_1, \dots x_k, y_1 \dots y_m \mid r_i(x), y_j w_j(x)^{-1} \rangle =$$

$$= \langle x_1, \dots x_k, y_1 \dots y_m \mid r_i(x), y_j w_j(x)^{-1}, x_s v_s(y)^{-1} \rangle =$$

$$= \langle x_1, \dots x_k, y_1 \dots y_m \mid r_i(v_s(y)), y_j w_j(v_s(y))^{-1}, x_s v_s(y)^{-1} \rangle =$$

$$= \langle y_1, \dots, y_m \mid r_i(v_s(y)), y_j w_j^{-1}(v_s(y)) \rangle$$

and lemma $\Rightarrow$ ok. $\square$